Mean field and cavity analysis for coupled oscillator networks

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(Received 22 May 2008; published 5 September 2008)

We study coupled oscillator spin systems on sparse, random graphs. In particular, we examine the recent conjecture of Ichinomiya on the equivalence of a sparsely connected oscillator network with ferromagnetic interactions to a fully connected network with disordered (i.e., randomly quenched) interactions. By restricting our investigation to a Hamiltonian case we can use the techniques of equilibrium statistical mechanics to compare these two models analytically including phase diagrams and the calculation of order parameters in the ordered phase. We complete our investigation by performing some Monte Carlo simulations to compare our theoretical predictions against.

DOI: 10.1103/PhysRevE.78.036106 PACS number(s): 89.75.Hc, 89.75.Fb, 05.45.Xt, 05.65.+b

I. INTRODUCTION

In recent years, much attention has been paid to large, multiparticle interacting systems where the interaction topology is described by a sparse random network [1–4]. By this we mean that each node or variable in the network interacts with a random subset of other nodes which is much smaller than the system size, and generally finite even in the infinite system size limit.

This current enthusiasm has been fueled by an abundance of systems (e.g., social networks [5], biological networks [6,7], neural networks [8], error-correcting codes [9], and optimization problems [10,11]) whose interactions are of this sparse, random type. The Kuramoto model [12] of coupled phase oscillators has provided a wealth of interesting models [13] describing synchronization phenomena. One important question has been how the topology of the network influences synchronization [14] which has been investigated using both numerical and theoretical techniques [15–18]. A recent paper by Ichinomiya [19] shed some light on this issue. By using the path-integral formalism [20,21] he was able to develop a link between a sparse random network of coupled oscillators with uniform interactions and a globally coupled (i.e., mean-field-type all-to-all interactions between the oscillators) network with quenched random interactions. This result can be seen heuristically by the method of moments which we describe briefly in Sec. II, although Ichinomiya showed it via an expansion in the average connectivity. The expansion and link between the two models was tested in [19] by the use of Monte Carlo techniques specifically looking at the Kuramoto transition. Our main result is showing that by restricting ourselves to a Hamiltonian system of coupled oscillators we can solve both the sparse random network (along the lines of [17,18]) and the fully connected random interaction network (along the lines of, e.g., [22,23] or see [24] for a more complete introduction to the replica theory). Since we have an analytic solution to these models we are able to compare them at a higher degree of resolution than simulations are able to provide, to see how well the fully connected (and hence simpler) system agrees with the sparse system and to quantify the magnitude of any difference in, e.g., the transition temperatures or the order parameters in the ordered phase (where the oscillators are entrained into one overall cycle) [25]. In particular, we show that in the two systems the transition temperature between entrained and disordered (paramagnetic) phases agree asymptotically as the connectivity parameter tends to infinity, whereas the behaviors are quite different for smaller values of the connectivity parameter. A spin-glass (disordered) low-temperature phase exists only in the fully connected system. In other words, Ichinomiya's conjecture holds asymptotically but is not exact for oscillators with a finite number of neighbors.

We define the two models in Sec. II and give a brief mean-field argument as to why they should behave similarly, in Sec. III we use replica theory to solve the fully connected network of oscillators and in Sec. IV we use the cavity approach to solve the sparse network of oscillators. We compare and contrast the two solutions in Sec. V showing some results of our analysis and some simulation results.

II. MODEL DEFINITIONS

We study a system of N coupled phase oscillators as introduced by Kuramoto [12]. In this model, each oscillator has a definite amplitude, and the state of a given oscillator is described by its phase $\phi \in \mathbb{R}$. The evolution equation for the phase ϕ_i of the ith oscillator is given by

$$\frac{d}{dt}\phi_i = \omega_i + \sum_{j \neq i} J_{ij} \sin(\phi_j - \phi_i) + \eta_i, \tag{1}$$

where $\eta_i(t)$ is the Gaussian white noise process with variance $2T$,

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$$\langle \eta_i(t)\eta_j(t')\rangle = 2T\delta_{ij}\delta(t-t'). \quad (2)$$

The solution of the model (1) in the thermodynamic limit (i.e., as $N\to\infty$) will, of course, depend to a great extent on the values taken by $\{\omega_i, J_{ij}\}$, or on their statistics if they are taken to be random variables, and is generally a highly nontrivial problem, although progress has been made in some cases, e.g., [26].

The archetypal sparse random network is the Erdös-Rényi random graph. For uniform ferromagnetic interactions between the spins a graph of this type can be generated by specifying

$$\text{model ER,} \quad P(J_{ij}) = \frac{c}{N}\delta_{J_{ij},1} + \left(1-\frac{c}{N}\right)\delta_{J_{ij},0} \quad \forall \; i<j, \quad (3)$$

where c is a finite constant and the interactions are taken to be symmetric. In the large system limit each oscillator is connected to a random number of other oscillators, this number is a Poisson distributed random variable with parameter c. So, the average number of oscillators it is connected to is c and the variance of the number is also c. As c increases, the Poisson (c) distribution is increasingly well approximated by a Gaussian distribution with mean c and variance c (with the distribution being totally specified by its mean and variance). Thus, along the lines of a mean-field-type argument we can write the following:

$$\left\langle \sum_{j\neq i} J_{ij}\sin(\phi_j-\phi_i)\right\rangle \approx \sum_{j\neq i}\langle J_{ij}\rangle\langle \sin(\phi_j-\phi_i)\rangle \quad (4)$$

$$= \frac{c}{N}\sum_{j\neq i}\langle \sin(\phi_j-\phi_i)\rangle. \quad (5)$$

Hence, the average field due to ferromagnetic connections with c other oscillators is replaced by a connection of strength c/N with all the other oscillators. Similarly, the fluctuations given by a differing number of connections per oscillator (according to the Poisson distribution) can be replaced by a disordered interaction with all other oscillators,

$$\text{model FC,} \quad J_{ij} = \frac{c}{N} + \sqrt{\frac{c}{N}}z_{ij} \quad \forall \; i<j, \quad (6)$$

where $z_{ij}\sim\mathcal{N}(0,1)$. This satisfies $\langle \Sigma_j J_{ij}\rangle = c$ and $\mathrm{var}(\Sigma_j J_{ij}) = c$. So the first two moments of the interactions for the fully connected (FC) model agree with those for the Erdös-Rényi (ER) random graph model. We have included the above argument on the relationship between the two models for completeness, for a more detailed analysis using perturbation theory applied within the path-integral formalism we refer the interested reader to [19].

Given the definition of the interactions (3) or even (6) the exact analysis of (1) is still a challenging problem. In this paper, in order to reach an analytical result, in the following we restrict ourselves to the case where $\omega_i=\omega$ for any i. Then, without loss of generality, we may assume $\omega=0$. Then Eq. (1) can be rewritten as

$$\frac{d}{dt}\phi_i = -\frac{\partial}{\partial \phi_i}H + \eta_i, \quad (7)$$

$$H(\boldsymbol{\phi}) = -\sum_{i<j} J_{ij}\cos(\phi_i-\phi_j). \quad (8)$$

Thus, the specification of a homogeneous driving force ω allow the model to be written as a Hamiltonian system. In the following two sections we use the techniques of equilibrium statistical mechanics and find analytic solutions to the equilibrium behavior of these two models so that they can be compared in Sec. V.

III. FULLY CONNECTED COUPLED OSCILLATOR NETWORK

Models of the form (7) and (8) subject to the definition (6) have been studied for some time [22,24] using the replica approach so we will only outline the method of solution. The replica approach allows one to calculate the disorder-averaged free energy per spin $\bar{f} = -\lim_{N\to\infty}(\beta N)^{-1}\overline{\ln Z}$ for model FC. It is given by

$$\bar{f} = -\lim_{N\to\infty}\lim_{n\to 0}(\beta N n)^{-1}\ln\int d\boldsymbol{\phi}^1\cdots d\boldsymbol{\phi}^n\,\overline{e^{-\beta\Sigma_\alpha H(\boldsymbol{\phi}^\alpha)}}, \quad (9)$$

where $\boldsymbol{\phi}^\alpha = (\phi_1^\alpha,\ldots,\phi_N^\alpha)$, $\alpha=1,\ldots,n$, and H is defined in (8) with the interactions $\{J_{ij}\}$ specified by (6). The averages over the quenched disorder variables $\{J_{ij}\}$ now amount to performing Gaussian averages. These can be performed with the resultant free energy being expressed as a saddle-point integral, thus it is given as an extremum,

$$\bar{f} = -\lim_{N\to\infty}\lim_{n\to 0}(\beta n)^{-1}\,\mathrm{extr}(\Phi+\Psi), \quad (10)$$

$$\begin{aligned}\Phi &= i\sum_\alpha(\hat{m}_c^\alpha m_c^\alpha + \hat{m}_s^\alpha m_s^\alpha) + i\sum_{\alpha\leq\beta}(\hat{q}_{cc}^{\alpha\beta}q_{cc}^{\alpha\beta} + \hat{q}_{ss}^{\alpha\beta}q_{ss}^{\alpha\beta} + \hat{q}_{cs}^{\alpha\beta}q_{cs}^{\alpha\beta})\\ &\quad + \frac{c\beta}{2}\sum_\alpha[(m_c^\alpha)^2 + (m_s^\alpha)^2]\\ &\quad + \frac{d\beta^2}{4}\sum_{\alpha\beta}[(q_{cc}^{\alpha\beta})^2 + (q_{ss}^{\alpha\beta})^2 + (q_{sc}^{\alpha\beta})^2 + (q_{cs}^{\alpha\beta})^2],\end{aligned} \quad (11)$$

$$\Psi = \ln\int d\boldsymbol{\phi}\, e^{-i\Sigma_\alpha[\hat{m}_c^\alpha\cos(\phi^\alpha)+\hat{m}_s^\alpha\sin(\phi^\alpha)] - i\Sigma_{\alpha\leq\beta}[\hat{q}_{cc}^{\alpha\beta}\cos(\phi^\alpha)\cos(\phi^\beta) + \hat{q}_{ss}^{\alpha\beta}\sin(\phi^\alpha)\sin(\phi^\beta) + \hat{q}_{cs}^{\alpha\beta}\cos(\phi^\alpha)\sin(\phi^\beta)]}. \quad (12)$$

Assuming that the above saddle point is replica symmetric, rotating variables in the complex plane and eliminating superfluous order parameters, the disorder-averaged free energy per spin can be written in the more compact form,

$$f = -\frac{c\beta}{4}(q_{cc}^2 + q_{ss}^2 + 2q_{sc}^2 - 1) - \frac{c}{2}(m_c^2 + m_s^2) - \frac{c\beta}{2}[Q_{cc}(Q_{cc} - 1) + Q_{sc}^2] + \frac{1}{\beta}\int DxDy \ln \int d\psi M(\psi|x,y) \tag{13}$$

with the effective measure

$$M(\psi|x,y) = e^{c\beta m_c \cos(\psi) + c\beta m_s \sin(\psi) + (1/2)c\beta^2(Q_{cc}-q_{cc})\cos^2(\psi) + (1/2)c\beta^2(1-Q_{cc}-q_{ss})\sin^2(\psi)}$$
$$\times e^{c\beta^2(Q_{sc}-q_{sc})\sin(\psi)\cos(\psi) + \beta x\sqrt{c(q_{cc}q_{ss}-q_{sc}^2)/q_{ss}}\cos(\psi) + \beta y\sqrt{c}[(q_{sc}/\sqrt{q_{ss}})\cos(\psi) + \sqrt{q_{ss}}\sin(\psi)]}. \tag{14}$$

The order parameters themselves must be solved from the self-consistent equations

$$m_c = [\langle\cos(\psi)\rangle], \quad m_s = [\langle\sin(\psi)\rangle],$$
$$Q_{cc} = [\langle\cos^2(\psi)\rangle], \quad Q_{sc} = [\langle\cos(\psi)\sin(\psi)\rangle],$$
$$q_{cc} = [\langle\cos(\psi)\rangle^2], \quad q_{ss} = [\langle\sin(\psi)\rangle^2],$$
$$q_{sc} = [\langle\sin(\psi)\rangle\langle\cos(\psi)\rangle], \tag{15}$$

where the averages in the above equations are given by

$$[\cdots] = \int DxDy \cdots , \tag{16}$$

$$\langle\cdots\rangle = \frac{\int d\psi M(\psi|x,y)\cdots}{\int d\psi M(\psi|x,y)}. \tag{17}$$

One of the interesting characteristics of the model is the phase diagram, which can be explored by locating the critical temperature using a bifurcation analysis. In the high-temperature phase $\beta \to 0$ we have $m_c = m_s = 0$ and $q_{cc} = q_{ss} = q_{sc} = Q_{sc} = 0$ while $Q_{cc} = \frac{1}{2}$. The orthogonality of our order parameters (when integrated over $[0, 2\pi]$) is useful as the bifurcation matrix is diagonal and thus the correct result is obtained by considering each order parameter term by term. The magnetization terms are straightforward as we find by expanding for small m_c, m_s that

$$m_c = [\langle\cos(\psi)\rangle] = \int \frac{d\psi}{2\pi}\cos(\psi)[c\beta m_c \cos(\psi)] = \frac{c\beta}{2}m_c \tag{18}$$

and as the calculation for m_s is identical the critical temperature $T_{m_c} = T_{m_s} = c/2$. Now, $T_{Q_{cs}}$ follows similarly via

$$Q_{cs} = [\langle\cos(\psi)\sin(\psi)\rangle]$$
$$= \int \frac{d\psi}{2\pi}\cos(\psi)\sin(\psi)[c\beta^2 Q_{cs}\sin(\psi)\cos(\psi)]$$
$$= \frac{c\beta^2}{8}Q_{cs} \tag{19}$$

which gives $T_{Q_{cs}} = \sqrt{c/8}$. Further, q_{cc} can also be found along similar lines,

$$q_{cc} = [\langle\cos(\psi)\rangle^2]$$
$$= \int DxDy\left(\int \frac{d\psi}{2\pi}\cos(\psi)[1 + \beta x\sqrt{dq_{cc}}\cos(\psi) + y\sqrt{cq_{ss}}\sin(\psi) + \cdots]\right)^2$$
$$= \frac{\beta^2 c q_{cc}}{4} \tag{20}$$

and (as q_{ss} and q_{cs} follow similarly) gives $T_{q_{cc}} = T_{q_{ss}} = \sqrt{c}/2$.

Our analysis leads to the conclusion that the critical temperatures for $P \to F$ and $P \to SG$ transitions are

$$P \to F, \quad T = \frac{c}{2}, \tag{21}$$

$$P \to SG, \quad T = \frac{\sqrt{c}}{2}, \tag{22}$$

note that the latter agrees with [22] (once we identify $\tilde{J}^2$ from their paper with c). For all $c > 1$ we enter a ferromagnetic phase at low temperatures (this is consistent with the percolation transition for model ER—below $c = 1$ there can be no macroscopically ordered phase). However, for $c < 1$ we have a spin-glass low-temperature phase in model FC (i.e., for $T < 0.5$) which cannot correspond at all to any state in a uniform bond dilute network of coupled oscillators.

IV. COUPLED OSCILLATOR NETWORK ON A RANDOM GRAPH

Models of the form (7) and (8) subject to the definition (3) have only been solved analytically relatively recently [17,18] (to which the reader can refer for more detailed solutions of the present model) due to the increased complexity of the analysis when each node of the network is only connected to a finite number of neighbors. This complexity manifests itself via the order parameters: Rather than having a few parameters that can fully characterise the system as in Sec. III, the self-consistent order parameter equations are written in terms of a disorder-averaged measure over probability distributions of individual oscillators,

$$W[\{P\}] = \lim_{N\to\infty} \overline{\left\langle \frac{1}{N}\sum_i P_i(\phi_i) \right\rangle}. \tag{23}$$

Thus, rather than having to solve for several parameters we have to solve for a measure over distributions, an altogether more complicated object. To see how this arises, we again invoke the replica method to calculate the disorder-averaged free energy as

$$\bar{f} = \lim_{n\to 0} \frac{1}{\beta n}\left(\frac{1}{2c} \int d\phi d\psi P(\phi)P(\psi)(e^{\beta\Sigma_\alpha \cos(\phi^\alpha-\psi^\alpha)} - 1) - \ln \int d\phi P(\phi) e^{c\int \psi P(\psi)[e^{\beta\Sigma_\alpha \cos(\phi^\alpha-\psi^\alpha)}-1]} \right), \tag{24}$$

where ϕ and ψ are both n-replicated vectors and the measure P is to be found from the self-consistent equation

$$P(\phi) = \frac{1}{\mathcal{N}} e^{c\int d\psi P(\psi) e^{\beta\Sigma_\alpha \cos(\phi^\alpha-\psi^\alpha)}}, \tag{25}$$

where $\mathcal{N}$ is just a normalization constant. The free energy in the above form is somewhat intractable and to make progress we require the usual replica-symmetric ansatz, which for the present model takes the form [18]

$$P(\phi) = \int \{dP\} W[\{P\}] \prod_\alpha P(\phi^\alpha) \tag{26}$$

which if we compare to the definition (23), we see that we can interpret the replica-symmetric ansatz as saying that the marginal equilibrium distribution for a given oscillator is the same in all replicas of the system (which is quite intuitive—since replica symmetry is normally closely linked to ergodicity and hence independence of initial system conditions [24]). Algebraically, the ansatz [26] allows one to take the limit $n\to 0$ giving the replica-symmetric disorder-averaged free energy f as

$$f = \frac{c}{2\beta}\int dP_1 dP_2 W[P_1]W[P_2] \ln \int d\phi_1 d\phi_2 P_1(\phi_1)P_2(\phi_2) e^{\beta\cos(\phi_1-\phi_2)} - \beta^{-1}\sum_{k\geq 0}\frac{e^{-c}c^k}{k!}\int \left(\prod_{\ell=1}^{k} dP_\ell W[P_\ell]\right) \ln \int d\phi \prod_{\ell=1}^{k} d\phi_\ell P(\phi_\ell) e^{\beta\Sigma_\ell \cos(\phi-\phi_\ell)}, \tag{27}$$

where the measure $W[\cdots]$ is found from the self-consistent equation

$$W[P] = \sum_{k\geq 0}\frac{e^{-c}c^k}{k!}\int \prod_{1\leq \ell\leq k} dP_\ell W[P_\ell] \delta_F \times \left(P(\phi) - \frac{1}{\mathcal{N}}\prod_\ell \int d\phi_\ell P_\ell(\phi_\ell) e^{\beta\cos(\phi-\phi_\ell)} \right), \tag{28}$$

where $\delta_F[\cdots]$ is a δ functional, returning zero when integrated over unless the argument is zero for the whole range of ϕ (except perhaps on sets of measure zero). Apart from in certain special cases (e.g., in the paramagnetic phase) is not clear how to treat the equation above analytically. However, a population dynamics approach can be used, with each member of the population somehow encoding a given distribution P, and the population as a whole converging towards W as both the size and the number of iterations of the dynamics increase.

The phase transitions of model ER can be found by first noting that $W[P]=\delta_F[P(\phi)-\frac{1}{2\pi}]$ is a solution of (28) corresponding to the high-temperature, paramagnetic state where the oscillators have no overall, nor individual alignment. Continuous transitions away from this state can be found using the so-called Guzai expansion [18] which is in essence perturbation expansion around the paramagnetic solution in small functions $\Delta(\phi)$ (i.e., one assumes that the magnitude of range of Δ is small)—$P(\phi)=\frac{1}{2\pi}+\Delta(\phi)$. Insertion of this ansatz into (28) and ignoring terms of second order or higher in Δ leads to the conclusion that an ordering transition in the oscillators away from the paramagnetic state occurs at

$$1 = \frac{cI_1(\beta)}{I_0(\beta)}, \tag{29}$$

where $I_n(\beta)$ is the nth modified Bessel function, $I_n(\beta) = \int_0^{2\pi}\frac{d\phi}{2\pi}\cos(n\phi)e^{z\cos(\phi)}$. Note that there is no spin-glass transition in this model, which is not surprising since there is no quenched bond disorder in the model (although there is quenched dilution disorder—the bonds are all positive, leading to forces tending to encourage synchronization, although any particular graph realization, out of the possible ensemble or random, dilute graphs, is random).

V. COMPARISON OF THE TWO MODELS

One could ask a variety of questions about how similar the two models we have considered are. We restrict ourselves to two main comparisons: What is the difference in the phase diagrams of model ER and model FC; and, how does the degree of synchronization compare between these two models in the ordered phase.

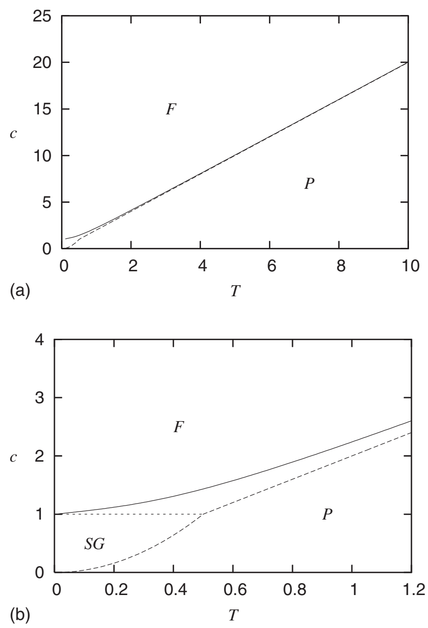

FIG. 1. (a) Phase diagram in T-c plane. Solid curve: Sparse model (model ER). Dotted curve: Fully connected model (model FC). (b) Enlargement of (a).

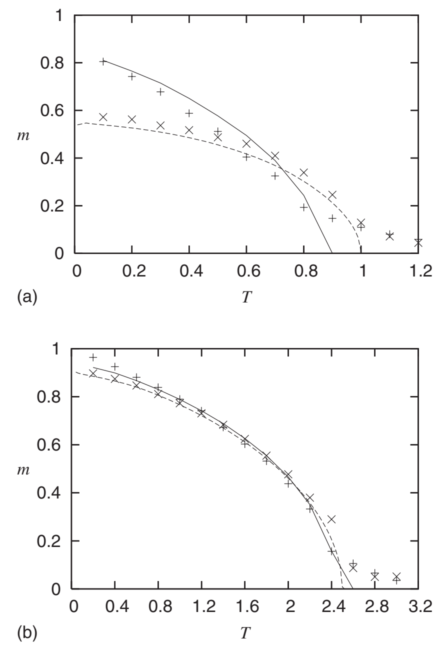

FIG. 2. T dependence of m. (a) $c=2$; (b) $c=5$. Curves; theoretical results. Symbols, numerical results of the Langevin equation (1) for $N=2000$. Solid curve and plus symbol, sparse model (model ER); dashed curve and cross, fully connected model (model FC).

In Fig. 1 we see the phase diagram of both models. Comparison of Eq. (21) with Eq. (29) shows that the transitions are not coincident, although an expansion for large c shows that they are in leading order. Indeed, even for values of c of order 1, the discrepancy between the transition temperatures is not very large, and would not necessarily be obvious from simulations except for quite large system sizes. Thus, not only do we see the agreement predicted by [19] but we can also measure quantitatively the degree of disagreement between the two models.

In Fig. 2 we show the value of a synchronization order parameter $m=\sqrt{m_c^2+m_s^2}$ where m_c and m_s are defined in Eq. (15) as we vary the temperature, for two different values of the connectivity. Again, as one would expect, the agreement between the models improves for larger values of c, and, as one may also expect, the disagreement is worst near the transition temperature where fluctuations and correlations will be largest. In Fig. 2, we display results by numerical simulations, too. The agreement between theoretical and numerical results is fairly well.

VI. CONCLUSION

In this paper we have investigated the similarity and differences of two different models for coupled oscillators with long-range coupling. The relationship between these models was pointed out by Ichinomiya [19] using path-integral analysis [21]. Ichinomiya having found this connection analyzed it using simulation techniques. This identity between the models is important due to the prevalence of random-graph-type network models in a variety of disciplines and the importance of the Kuramoto model in describing synchronization phenomena. We have sought to further the understanding of the relationship between these models, and the size and nature of disagreements between them by looking at a particular case where both models can be solved analytically, namely when they are both in equilibrium. This has allowed us to compare their phase diagrams and the value of synchronization order parameters in the ordered phase. This clarifies the parameter region where the two models behave differently. In particular, when c is less than 1, a spin-glass low-temperature phase appears for the fully connected system. In contrast, the sparsely connected system does not percolate for c less than 1 and so the equilibrium state is the paramagnetic phase for the sparsely connected system.

ACKNOWLEDGMENT

The authors are grateful to A. C. C. Coolen for valuable discussions.

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